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21. Materials and methods are available as supporting material on *Science* Online.
22. A. Heger, N. Langer, *Astron. Astrophys.* **334**, 210 (1998).
23. A. P. Crotts, S. R. Heathcote, *Nature* **350**, 683 (1991).
24. J. Xu, A. Crotts, W. Kunkel, *Astrophys. J.* **451**, 806 (1995).
25. B. Sugerman, A. Crotts, W. Kunkel, S. Heathcote, S. Lawrence, *Astrophys. J.* **627**, 888 (2005).
26. N. Soker, *Astrophys. J.*, in press; preprint available online (http://xxx.lanl.gov/abs/astro-ph/0610655)
27. N. Panagia *et al.*, *Astrophys. J.* **459**, L17 (1996).

28. The authors would like to thank L. Nelson for providing access to the Bishop/Sherbrooke Beowulf cluster (Elix3) which was used to perform the interacting winds calculations. The binary merger calculations were performed on the UK Astrophysical Fluids Facility. T.M. acknowledges support from the Research Training Network "Gamma-Ray Bursts: An Enigma and a Tool" during part of this work.

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Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture

Peter J. Lu[1]* and Paul J. Steinhardt[2]

The conventional view holds that girih (geometric star-and-polygon, or strapwork) patterns in medieval Islamic architecture were conceived by their designers as a network of zigzagging lines, where the lines were drafted directly with a straightedge and a compass. We show that by 1200 C.E. a conceptual breakthrough occurred in which girih patterns were reconceived as tessellations of a special set of equilateral polygons ("girih tiles") decorated with lines. These tiles enabled the creation of increasingly complex periodic girih patterns, and by the 15th century, the tessellation approach was combined with self-similar transformations to construct nearly perfect quasi-crystalline Penrose patterns, five centuries before their discovery in the West.

Girih patterns constitute a wide-ranging decorative idiom throughout Islamic art and architecture (*1*–*6*). Previous studies of medieval Islamic documents describing applications of mathematics in architecture suggest that these girih patterns were constructed by drafting directly a network of zigzagging lines (sometimes called strapwork) with the use of a compass and straightedge (*3*, *7*). The visual impact of these girih patterns is typically enhanced by rotational symmetry. However, periodic patterns created by the repetition of a single "unit cell" motif can have only a limited set of rotational symmetries, which western mathematicians first proved rigorously in the 19th century C.E.: Only two-fold, three-fold, four-fold, and six-fold rotational symmetries are allowed. In particular, five-fold and 10-fold symmetries are expressly forbidden (*8*). Thus, although pentagonal and decagonal motifs appear frequently in Islamic architectural tilings, they typically adorn a unit cell repeated in a pattern with crystallographically allowed symmetry (*3*–*6*).

Although simple periodic girih patterns incorporating decagonal motifs can be constructed using a "direct strapwork method" with a straightedge and a compass (as illustrated in Fig. 1, A to D), far more complex decagonal patterns also occur in medieval Islamic architecture. These complex patterns can have unit cells containing hundreds of decagons and may repeat the same decagonal motifs on several length scales. Individually placing and drafting hundreds of such decagons with straightedge and compass would have been both exceedingly cumbersome and likely to accumulate geometric distortions, which are not observed.

On the basis of our examination of a large number of girih patterns decorating medieval Islamic buildings, architectural scrolls, and other forms of medieval Islamic art, we suggest that by 1200 C.E. there was an important breakthrough in Islamic mathematics and design: the discovery of an entirely new way to conceptualize and construct girih line patterns as decorated tessellations using a set of five tile types, which we call "girih tiles." Each girih tile is decorated with lines and is sufficiently simple to be drawn using only mathematical tools documented in medieval Islamic sources. By laying the tiles edge-to-edge, the decorating lines connect to form a continuous network across the entire tiling. We further show how the girih-tile approach opened the path to creating new types of extraordinarily complex patterns, including a nearly perfect quasi-crystalline Penrose pattern on the Darb-i Imam shrine (Isfahan, Iran, 1453 C.E.), whose underlying mathematics were not understood for another five centuries in the West.

As an illustration of the two approaches, consider the pattern in Fig. 1E from the shrine of Khwaja Abdullah Ansari at Gazargah in Herat, Afghanistan (1425 to 1429 C.E.) (*3*, *9*), based on a periodic array of unit cells containing a common decagonal motif in medieval Islamic architecture, the 10/3 star shown in Fig. 1A (see fig. S1 for additional examples) (*1*, *3*–*5*, *10*). Using techniques documented by medieval Islamic mathematicians (*3*, *7*), each motif can be drawn using the direct strapwork method (Fig. 1, A to D). However, an alternative geometric construction can generate the same pattern (Fig. 1E, right). At the intersections between all pairs of line segments not within a 10/3 star, bisecting the larger 108° angle yields line segments (dotted red in the figure) that, when extended until they intersect, form three distinct polygons: the decagon decorated with a 10/3 star line pattern, an elongated hexagon decorated with a bat-shaped line pattern, and a bowtie decorated by two opposite-facing quadrilaterals. Applying the same procedure to a ~15th-century pattern from the Great Mosque of Nayriz, Iran (fig. S2) (*11*) yields two additional polygons, a pentagon with a pentagonal star pattern, and a rhombus with a bowtie line pattern. These five polygons (Fig. 1F), which we term "girih tiles," were used to construct a wide range of patterns with decagonal motifs (fig. S3) (*12*). The outlines of the five girih tiles were also drawn in ink by medieval Islamic architects in scrolls drafted to transmit architectural practices, such as a 15th-century Timurid-Turkmen scroll now held by the Topkapi Palace Museum in Istanbul (Fig. 1G and fig. S4) (*2*, *13*), providing direct historical documentation of their use.

The five girih tiles in Fig. 1F share several geometric features. Every edge of each polygon has the same length, and two decorating lines intersect the midpoint of every edge at 72° and 108° angles. This ensures that when the edges of two tiles are aligned in a tessellation, decorating lines will continue across the common boundary without changing direction (*14*). Because both line intersections and tiles only contain angles that are multiples of 36°, all line segments in the final girih strapwork pattern formed by girih-tile decorating lines will be parallel to the sides of the regular pentagon; decagonal geometry is thus enforced in a girih pattern formed by the tessellation of any combination of girih tiles. The tile decorations have different internal rotational symmetries: the decagon, 10-fold symmetry; the pentagon, five-fold; and the hexagon, bowtie, and rhombus, two-fold.

Tessellating these girih tiles provides several practical advantages over the direct strapwork method, allowing simpler, faster, and more accurate execution by artisans unfamiliar with their mathematical properties. A few full-size girih tiles could serve as templates to help position decorating lines on a building surface, allowing rapid, exact pattern generation. More-

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over, girih tiles minimize the accumulation of angular distortions expected in the manual drafting of individual 10/3 stars, with concomitant errors in sizing, position, and orientation.

Girih tiles further enable the construction of periodic decagonal-motif patterns that do not arise naturally from the direct strapwork method. One class of such patterns repeats pentagonal motifs but entirely lacks the 10/3 stars that establish the initial decagonal angles needed for direct drafting with straightedge and compass. Patterns of this type appear around 1200 C.E. on Seljuk buildings, such as the Mama Hatun Mausoleum in Tercan, Turkey (1200 C.E.; Fig. 2A) (*5*, *15*, *16*), and can be created easily by tessellating bowtie and hexagon girih tiles to create perfect pentagonal motifs, even in the absence of a decagon star (i.e., lacking decagon girih tiles; see fig. S5). Even more compelling evidence for the use of girih tiles occurs on the walls of the Gunbad-i Kabud in Maragha, Iran (1197 C.E.) (*11*, *17*, *18*), where seven of eight exterior wall panels on the octagonal tomb tower are filled with a tiling of decagons, hexagons, bowties, and rhombuses (Fig. 2, B and C). Within each wall panel, the decagonal pattern does not repeat; rather, the unit cell of this periodic tiling spans the length of two complete panels (fig. S6). The main decorative raised brick pattern follows the girih-tile decorating lines of Fig. 1F. However, a second set of smaller decorative lines conforms to the internal rotational symmetry of each individual girih tile without adhering to pentagonal angles (Fig. 2, C and D): Within each region occupied by a hexagon, bowtie, or rhombus, the smaller line decoration has a two-fold, not five-fold, rotational symmetry, and therefore could not have been generated using the direct strapwork method. By contrast, constructing both patterns is straightforward with girih tiles. Two sets of line decoration were applied to each girih tile: the standard line decoration of Fig. 1F, and a second, nonpentagonal set of motifs with an overall two-fold symmetry (Fig. 2, C and D). The girih tiles were then tessellated, with the regular line pattern expressed in large raised brick on the tower and the second set of lines expressed in smaller bricks. The dual-layer nature of line patterns on the Maragha tower thus adds strong evidence that the pattern was generated by tessellating with the girih tiles in Fig. 1F.

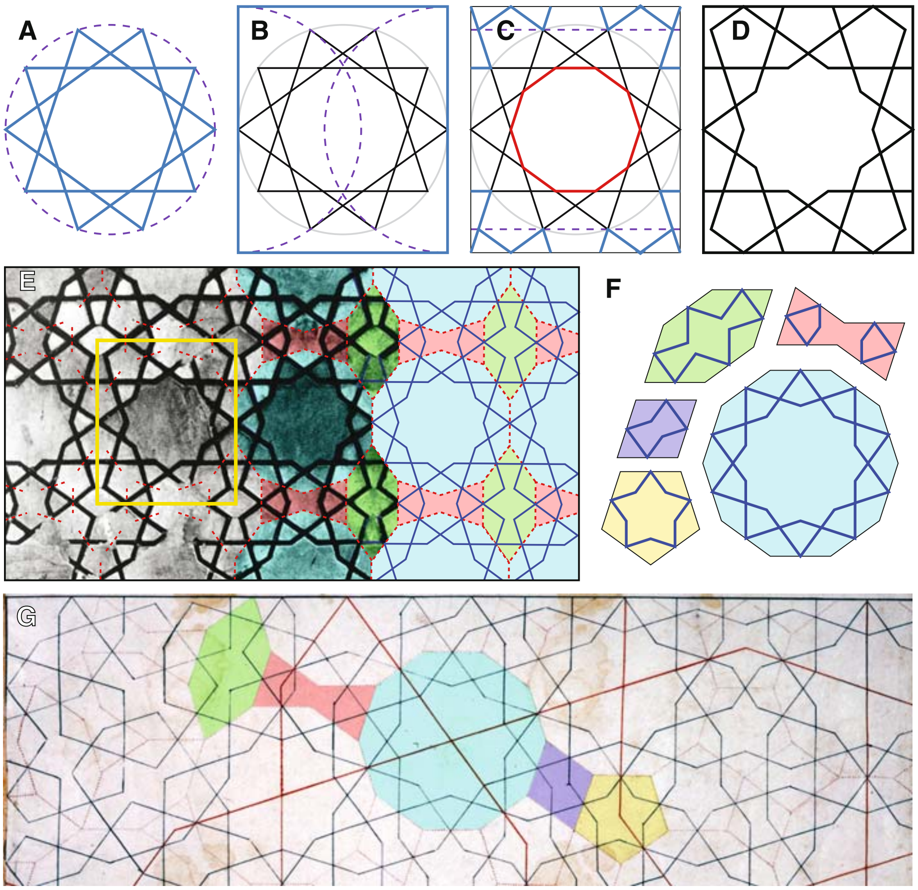

Fig. 1. Direct strapwork and girih-tile construction of 10/3 decagonal patterns. (**A** to **D**) Generation of a common 10/3 star pattern by the direct strapwork method. (A) A circle is divided equally into 10, and every third vertex is connected by a straight line to create the 10/3 star that (B) is centered in a rectangle whose width is the circle's diameter. In each step, new lines drafted are indicated in blue, lines to be deleted are in red, and purple construction lines not in the final pattern are in dashed purple. (**E**) Periodic pattern at the Timurid shrine of Khwaja Abdullah Ansari at Gazargah in Herat, Afghanistan (1425 to 1429 C.E.), where the unit cell pattern (D) is indicated by the yellow rectangle. The same pattern can be obtained by tessellating girih tiles (overlaid at right). (**F**) The complete set of girih tiles: decagon, pentagon, hexagon, bowtie, and rhombus. (**G**) Ink outlines for these five girih tiles appear in panel 28 of the Topkapi scroll, where we have colored one of each girih tile according to the color scheme in (F).

Perhaps the most striking innovation arising from the application of girih tiles was the use of self-similarity transformation (the subdivision of large girih tiles into smaller ones) to create overlapping patterns at two different length scales, in which each pattern is generated by the same girih tile shapes. Examples of subdivision can be found in the Topkapi scroll (e.g., Fig. 1G; see also fig. S4A) and on the Friday Mosque (*17*) and Darb-i Imam shrine (1453 C.E.) (*2*, *9*, *19*) in Isfahan, Iran. A spandrel from the Darb-i Imam shrine is shown in Fig. 3A. The large, thick, black line pattern consisting of a handful of decagons and bowties (Fig. 3C) is subdivided into the smaller pattern, which can also be perfectly generated by a tessellation of 231 girih tiles (Fig. 3B; line decoration of Fig. 1F filled in with solid color here). We have identified the subdivision rule used to generate the Darb-i Imam spandrel pattern (Fig. 3, D and E), which was also used on other patterns on the Darb-i Imam shrine and Isfahan Friday Mosque (fig. S7).

A subdivision rule, combined with decagonal symmetry, is sufficient to construct perfect quasi-crystalline tilings—patterns with infinite perfect quasi-periodic translational order and crystallographically forbidden rotational symmetries, such as pentagonal or decagonal—which mathematicians and physicists have come to understand only in the past 30 years (*20*, *21*). Quasi-periodic order means that distinct tile shapes repeat with frequencies that are incommensurate; that is, the ratio of the frequencies cannot be expressed as a ratio of integers. By having quasi-periodicity rather than periodicity, the symmetry constraints of conventional crystallography can be violated, and it is possible to have pentagonal motifs that join together in a pattern with overall pentagonal and decagonal symmetry (*21*).

The most famous example of a quasi-crystalline tiling is the Penrose tiling (*20*, *22*), a two-tile tessellation with long-range quasi-periodic translational order and five-fold symmetry. The Penrose tiles can have various shapes. A convenient choice for comparison with medieval Islamic architectural decoration is the kite and dart shown on the left side of Fig. 4, A and B. As originally conceived by Penrose in the 1970s, the tilings can be constructed either by "matching rules" or by self-similar subdivisions. For the matching rules, the kite and dart can each be decorated with red and blue stripes (Fig. 4, A and B); when tiles are placed so that the stripes continue uninterrupted, the only possible close-packed arrangement is a five-fold symmetric quasi-crystalline pattern in which the kites and darts repeat with frequencies whose ratio is irrational, namely, the golden ratio $\tau \equiv (1 + \sqrt{5})/2 \approx 1.618$. We see no evidence that Islamic designers used the matching-rule approach. The second approach is to repeatedly subdivide kites and darts into smaller kites and darts, according to the rules shown in Fig. 4, A and B. This self-similar subdivision of large tiles into small tiles can be expressed in terms of a transformation matrix whose eigenvalues are irrational, a signature of quasi-periodicity; the eigenvalues represent the ratio of tile frequencies in the limit of an infinite tiling (*23*).

Our analysis indicates that Islamic designers had all the conceptual elements necessary to produce quasi-crystalline girih patterns using the self-similar transformation method: girih tiles, decagonal symmetry, and subdivision. The pattern on the Darb-i Imam shrine is a remarkable example of how these principles were applied. Using the self-similar subdivision of large girih tiles into small ones shown in Fig. 3, D and E, an arbitrarily large Darb-i Imam pattern can be constructed. The asymptotic ratio of hexagons to bowties approaches the golden ratio τ (the same ratio as kites to darts in a Penrose tiling), an irrational ratio that shows explicitly that the pattern is quasi-periodic.

Moreover, the Darb-i Imam tile pattern can be mapped directly into Penrose tiles following the prescription for the hexagon, bowtie (*22*), and decagon given in Fig. 4, C to E. Using these substitutions, both the large (Fig. 3C) and small (Fig. 3B) girih-tile patterns on the Darb-i Imam can be mapped completely into Penrose tiles (fig. S8). Note that the mapping shown in Fig. 4, C to E, breaks the bilateral symmetries of the girih tiles; as a result, for an individual tile, there is a discrete number of choices for the mapping: 10 for the decagon, two each for hexagon and bowtie. Therefore, the mapping is completed by using this freedom to eliminate Penrose tile edge mismatches to the maximum degree possible. Note that, unlike previous comparisons in the literature between Islamic designs with decagonal motifs and Penrose tiles (*18*, *24*), the Darb-i Imam tessellation is not embedded in a periodic framework and can, in principle, be extended into an infinite quasi-periodic pattern.

Although the Darb-i Imam pattern illustrates that Islamic designers had all the elements needed to construct perfect quasi-crystalline patterns, we nonetheless find indications that the designers had an incomplete understanding of these elements. First, we have no evidence that they ever developed the alternative matching-rule approach. Second, there are a small number of tile mismatches, local imperfections in the Darb-i Imam tiling. These can be visualized by mapping the tiling into the Penrose tiles and identifying the mismatches. However, there are only a few of them—11 mismatches out of 3700 Penrose tiles—and every mismatch is point-like, removable with a local rearrangement of a few tiles without affecting the rest of the pattern (Fig. 4F and fig. S8). This is the kind of defect that an artisan could have made inadvertently in constructing or repairing a complex pattern. Third, the designers did not begin with a single girih tile, but rather with a small arrangement of large tiles that does not appear in the subdivided pattern. This arbitrary and unnecessary choice means that, strictly speaking, the tiling is not self-similar, although repeated application of the subdivision rule would nonetheless lead to the same irrational τ ratio of hexagons to bowties.

Our work suggests several avenues for further investigation. Although the examples we have studied thus far fall just short of being perfect quasi-crystals, there may be more interesting examples yet to be discovered, including perfectly quasi-periodic decagonal patterns. The subdivision analysis outlined above establishes a procedure for identifying quasi-periodic patterns

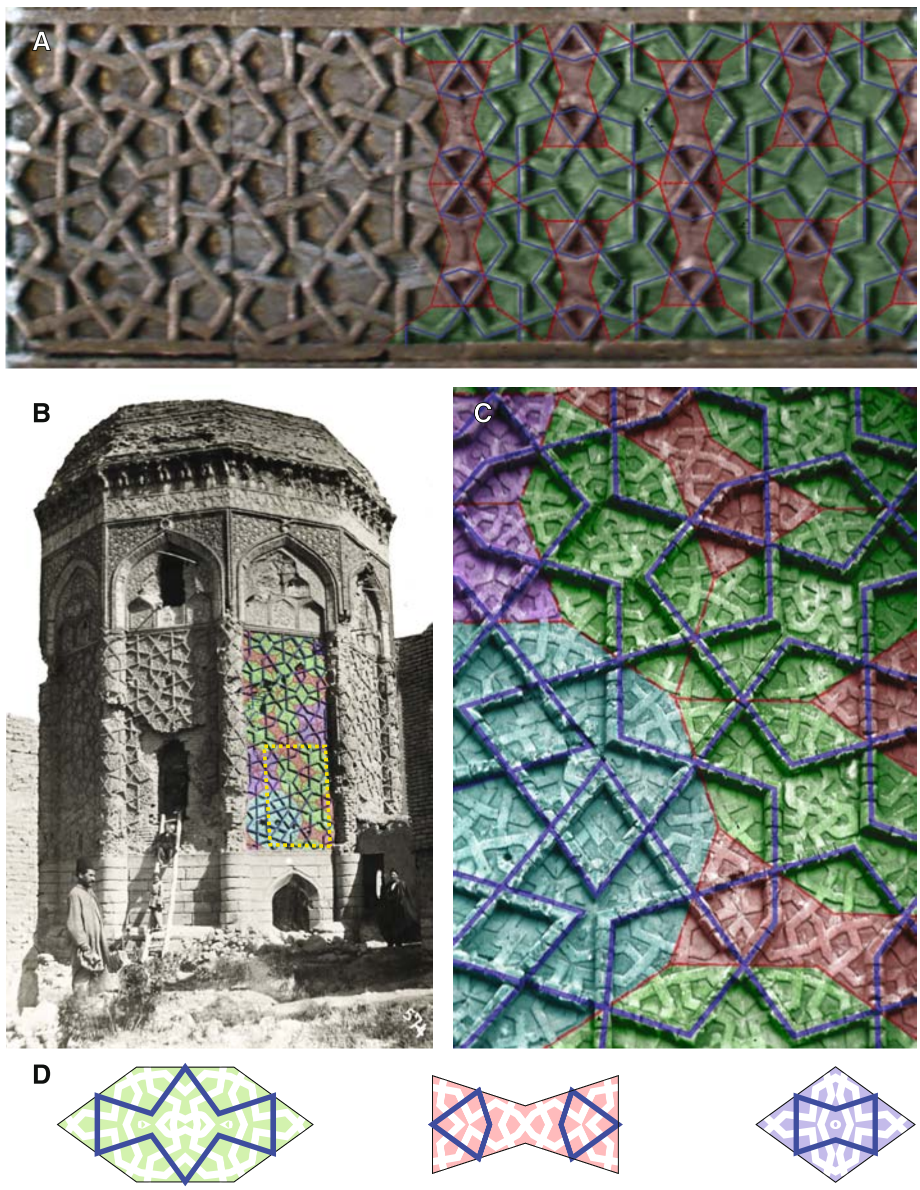

Fig. 2. (**A**) Periodic girih pattern from the Seljuk Mama Hatun Mausoleum in Tercan, Turkey (~1200 C.E.), where all lines are parallel to the sides of a regular pentagon, even though no decagon star is present; reconstruction overlaid at right with the hexagon and bowtie girih tiles of Fig. 1F. (**B**) Photograph by A. Sevruguin (~1870s) of the octagonal Gunbad-i Kabud tomb tower in Maragha, Iran (1197 C.E.), with the girih-tile reconstruction of one panel overlaid. (**C**) Close-up of the area marked by the dotted yellow rectangle in (B). (**D**) Hexagon, bowtie, and rhombus girih tiles with additional small-brick pattern reconstruction (indicated in white) that conforms not to the pentagonal geometry of the overall pattern, but to the internal two-fold rotational symmetry of the individual girih tiles.

and measuring their degree of perfection. Also, analogous girih tiles may exist for other noncrystallographic symmetries, and similar dotted tile outlines for nondecagonal patterns appear in the Topkapi scroll. Finally, although our analysis shows that complex decagonal tilings were being made by 1200 C.E., exactly when the shift from the direct strapwork to the girih-tile paradigm first occurred is an open question, as is the identity of the designers of these complex Islamic patterns, whose geometrical sophistication led the medieval world.

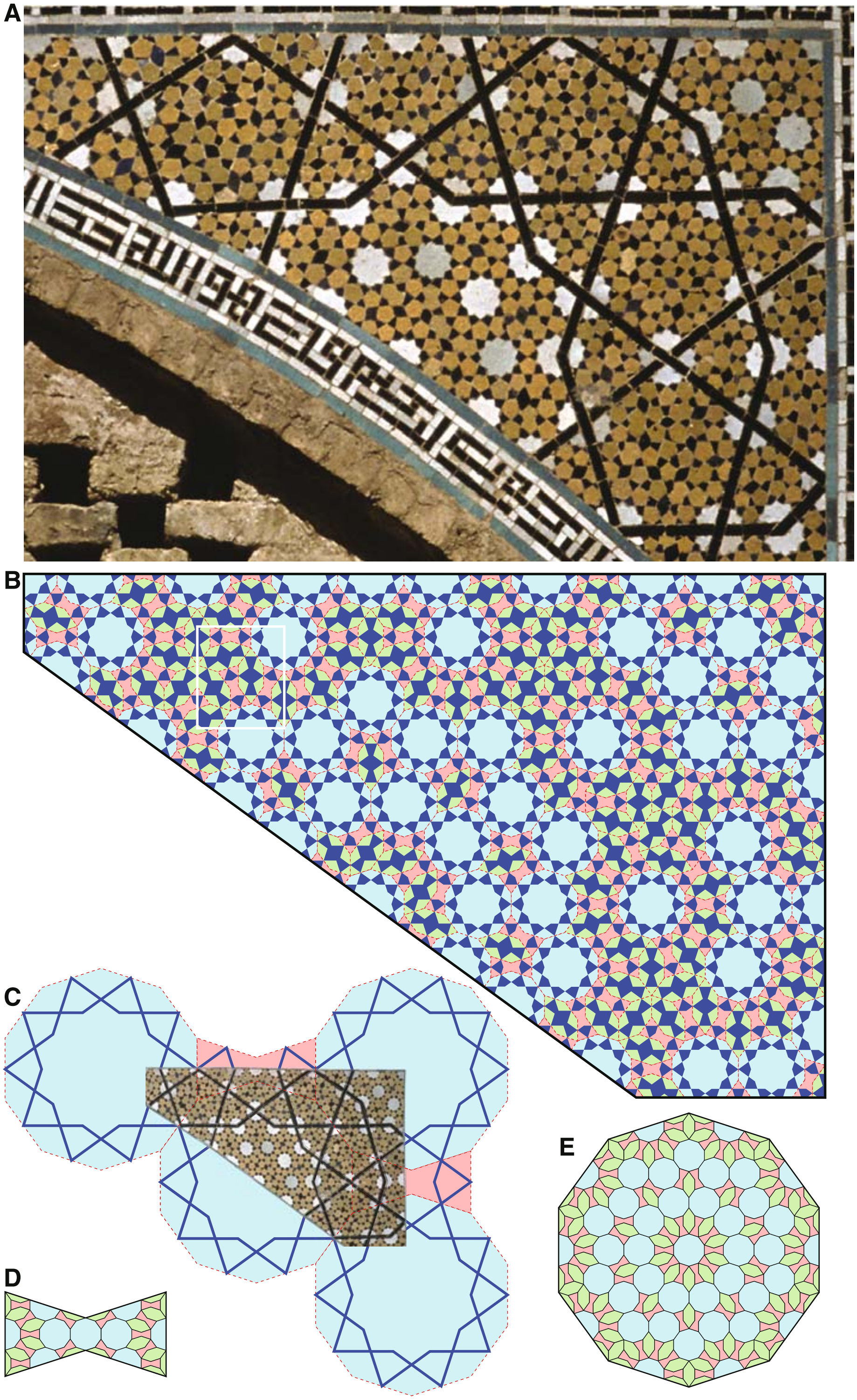

Fig. 3. Girih-tile subdivision found in the decagonal girih pattern on a spandrel from the Darb-i Imam shrine, Isfahan, Iran (1453 C.E.). (**A**) Photograph of the right half of the spandrel. (**B**) Reconstruction of the smaller-scale pattern using girih tiles where the blue-line decoration in Fig. 1F has been filled in with solid color. (**C**) Reconstruction of the larger-scale thick line pattern with larger girih tiles, overlaid on the building photograph. (**D** and **E**) Graphical depiction of the subdivision rules transforming the large bowtie (D) and decagon (E) girih-tile pattern into the small girih-tile pattern on tilings from the Darb-i Imam shrine and Friday Mosque of Isfahan.

References and Notes

1. J. Bourgoin, in *Les elements de l'art arabe; le trait des entrelacs* (Firmin-Didot, Paris, 1879), p. 176.
2. G. Necipoglu, *The Topkapi Scroll: Geometry and Ornament in Islamic Architecture* (Getty Center for the History of Art and the Humanities, Santa Monica, CA, 1995).
3. I. El-Said, A. Parman, in *Geometric Concepts in Islamic Art* (World of Islam Festival, London, 1976), pp. 85–87.
4. S. J. Abas, A. S. Salman, in *Symmetries of Islamic Geometrical Patterns* (World Scientific, Singapore, 1995), p. 95.
5. Y. Demiriz, in *Islam Sanatinda Geometrik Susleme* (Lebib Yalkin, Istanbul, 2000), pp. 27, 128–129.
6. G. Schneider, in *Geometrische Bauornamente der Seldschuken in Kleinasien* (Reichert, Wiesbaden, Germany, 1980), pp. 136–139, plate 3.
7. Abu'l-Wafa al-Buzjani's (940–998 C.E.) treatise *On the Geometric Constructions Necessary for the Artisan*, and an anonymous manuscript appended to a Persian translation of al-Buzjani and likely dating from the 13th century, *On Interlocks of Similar or Corresponding Figures* (*2*), document specific techniques for architecturally related mathematical constructions (*2*, *25*). The mathematical tools needed to construct the girih tiles are entirely contained in these two manuscripts—specifically, bisection, division of a circle into five equal parts, and cutting and rearrangement of paper tiles to create geometric patterns.
8. A. Bravais, *J. Ec. Polytech.* **33**, 1 (1850).
9. L. Golombek, D. Wilber, in *The Timurid Architecture of Iran and Turan* (Princeton Univ. Press, Princeton, NJ, 1988), pp. 246–250, 308–309, 384–386, 389, color plates IV, IXb, plates 46, 374.
10. Additional examples of this particular 10/3 decagonal pattern, shown in fig. S1: the Seljuk Congregational Mosque in Ardistan, Iran (~1160 C.E.) (*16*); the Timurid Tuman Aqa Mausoleum in the Shah-i Zinda complex in Samarkand, Uzbekistan (1405 C.E.) (*9*, *16*); the Darb-i Kushk shrine in Isfahan, Iran (1496 C.E.) (*2*, *9*, *17*); and the Mughal I'timad al-Daula Mausoleum in Agra, India (~1622 C.E.) (*28*).
11. R. Ettinghausen, O. Grabar, M. Jenkins-Madina, in *Islamic Art and Architecture 650–1250* (Yale Univ. Press, New Haven, CT, 2001), p. 109.
12. Additional architectural examples of patterns that can be reconstructed with girih tiles, shown in fig. S3: the Abbasid Al-Mustansiriyya Madrasa in Baghdad, Iraq (1227 to 1234 C.E.) (*26*); the Ilkhanid Uljaytu Mausoleum in Sultaniya, Iran (1304 C.E.) (*17*); the Ottoman Green Mosque in Bursa, Turkey (1424 C.E.) (*27*); and the Mughal I'timad al-Daula Mausoleum in Agra, India (~1622 C.E.) (*28*). Similar patterns also appear in the Mamluk Qurans of Sandal (1306 to 1315 C.E.) and of Aydughdi ibn Abdallah al-Badri (1313 C.E.) (*29*). Note that the girih-tile paradigm can make pattern design structure more clear. For example, all of the spandrels with decagonal girih patterns we have thus far examined (including Fig. 3C and figs. S2 and S3A) follow the same prescription to place decagons: Partial decagons are centered at the four external corners and on the top edge directly above the apex of the arch.
13. A similar convention was used to mark the same girih tiles in other panels (e.g., 28, 50, 52, and 62) in the Topkapi scroll (fig. S4) (*2*).
14. E. H. Hankin, *The Drawing of Geometric Patterns in Saracenic Art* (Government of India Central Publications Branch, Calcutta, 1925), p. 4.
15. This pattern type also occurs on the Great Mosque in Malatya, Turkey (~1200 C.E.) (*6*), and the madrasa in

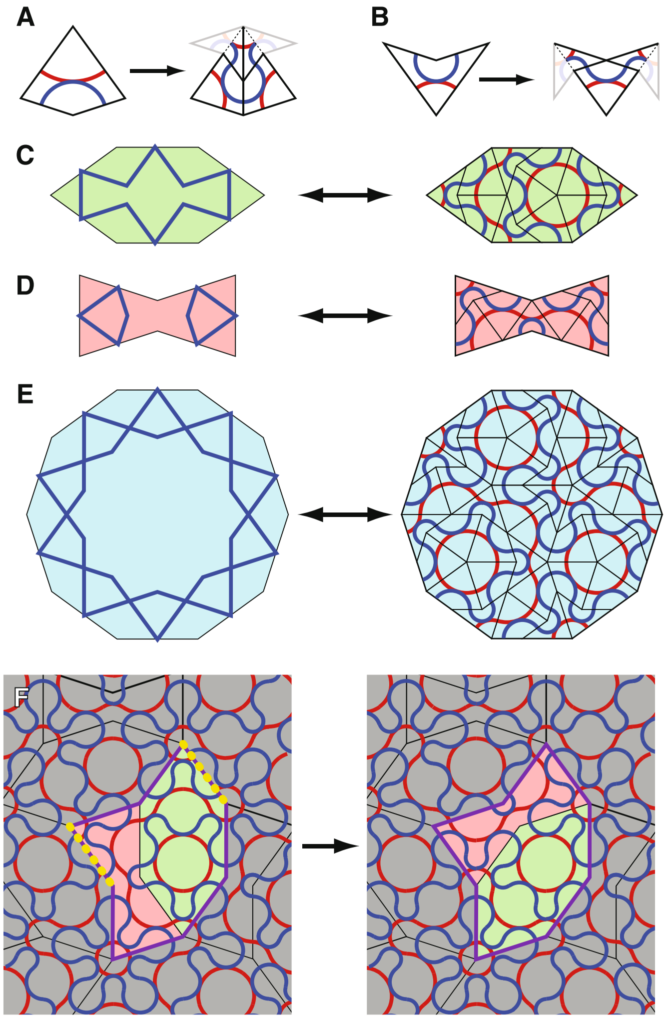

Fig. 4. (**A** and **B**) The kite (A) and dart (B) Penrose tile shapes are shown at the left of the arrows with red and blue ribbons that match continuously across the edges in a perfect Penrose tiling. Given a finite tiling fragment, each tile can be subdivided according to the "inflation rules" into smaller kites and darts (at the right of the arrows) that join together to form a perfect fragment with more tiles. (**C** to **E**) Mappings between girih tiles and Penrose tiles for elongated hexagon (C), bowtie (D), and decagon (E). (**F**) Mapping of a region of small girih tiles to Penrose tiles, corresponding to the area marked by the white rectangle in Fig. 3B, from the Darb-i Imam shrine. At the left is a region mapped to Penrose tiles following the rules in (C) to (E). The pair of colored tiles outlined in purple have a point defect (the Penrose edge mismatches are indicated with yellow dotted lines) that can be removed by flipping positions of the bowtie and hexagon, as shown on the right, yielding a perfect, defect-free Penrose tiling.

Zuzan, Iran (1219 C.E.) (*30*) (fig. S5), as well as on a carved wooden double door from a Seljuk building in Konya (~13th century C.E.), in the Museum of Islamic Art in Berlin (Inv. Nr. 1.2672).
16. D. Hill, O. Grabar, in *Islamic Architecture and Its Decoration, A.D. 800–1500* (Univ. of Chicago Press, Chicago, 1964), pp. 53, 62, 65, plates 38, 276, 346, 348.
17. S. P. Seherr-Thoss, *Design and Color in Islamic Architecture* (Smithsonian Institution, Washington, DC, 1968), plates 34–36, 40, 84, 90.
18. E. Makovicky, in *Fivefold Symmetry*, I. Hargittai, Ed. (World Scientific, Singapore, 1992), pp. 67–86.
19. J. F. Bonner, in *ISAMA/Bridges Conference Proceedings*, R. Sarhangi, N. Friedman, Eds. (Univ. of Granada, Granada, Spain, 2003), pp. 1–12.
20. R. Penrose, *Bull. Inst. Math. Appl.* **10**, 266 (1974).
21. D. Levine, P. J. Steinhardt, *Phys. Rev. Lett.* **53**, 2477 (1984).
22. M. Gardner, in *Penrose Tiles to Trapdoor Ciphers* (Freeman, New York, 1989), pp. 1–29.
23. D. Levine, P. J. Steinhardt, *Phys. Rev. B* **34**, 596 (1986).
24. A single figure, part of a geometric proof from *On Interlocks of Similar or Corresponding Figures*, has been related to the outlines of individual Penrose tiles, but there is no evidence whatsoever for tessellation (*31*). Makovicky has connected the Maragha Gunbad-i Kabud pattern in Fig. 2 with the Penrose tiling (*18*), but explicitly states (as we show in fig. S6) that the pattern is periodic, so by definition it cannot be a properly quasi-periodic Penrose tiling.
25. A. Ozdural, *Hist. Math.* **27**, 171 (2000).
26. H. Schmid, *Die Madrasa des Kalifen Al-Mustansir in Baghdad* (Zabern, Mainz, Germany, 1980), plates 15, 87.
27. G. Goodwin, *A History of Ottoman Architecture* (Thames and Hudson, London, 1971), pp. 58–65.
28. Y. Ishimoto, *Islam: Space and Design* (Shinshindo, Kyoto, 1980), plates 378, 380, 382.
29. D. James, *Qur'ans of the Mamluks* (Thames and Hudson, New York, 1988), pp. 54, 57–59.
30. R. Hillenbrand, *Islamic Architecture* (Columbia Univ. Press, New York, 1994), pp. 182–183.
31. W. K. Chorbachi, *Comp. Math. Appl.* **17**, 751 (1989).
32. We thank G. Necipoglu and J. Spurr, without whose multifaceted assistance this paper would not have been possible. We also thank R. Holod and K. Dudley/M. Eniff for permission to reproduce their photographs in Figs. 2C and 3A, respectively; C. Tam and E. Simon-Brown for logistical assistance in Uzbekistan; S. Siavoshi and A. Tafvizi for motivating the exploration of Isfahan's sights; and S. Blair, J. Bloom, C. Eisenmann, T. Lentz, and I. Winter for manuscript comments. Photographs in Fig. 2, A and B, and in the online figures courtesy of the Fine Arts Library, Harvard College Library. Supported by C. and F. Lu and by the Aga Khan Program for Islamic Architecture at Harvard University.

Supporting Online Material
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Figs. S1 to S8

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Ex Situ NMR in Highly Homogeneous Fields: ^{1}H Spectroscopy

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Portable single-sided nuclear magnetic resonance (NMR) magnets used for nondestructive studies of large samples are believed to generate inherently inhomogeneous magnetic fields. We demonstrated experimentally that the field of an open magnet can be shimmed to high homogeneity in a large volume external to the sensor. This technique allowed us to measure localized high-resolution proton spectra outside a portable open magnet with a spectral resolution of 0.25 part per million. The generation of these experimental conditions also simplifies the implementation of such powerful methodologies as multidimensional NMR spectroscopy and imaging.

Single-sided nuclear magnetic resonance (NMR) sensors have been used for over two decades to characterize arbitrarily large samples (*1*). In contrast to conventional NMR apparatus, where the sample must be adapted to fit into the bore of large superconducting magnets, single-sided NMR experiments use portable open magnets placed on one side of an object to detect NMR signals ex situ. This configuration is convenient for the nondestructive inspection of valuable objects from which fragmentary samples cannot be drawn, but it does not allow generation of the high and homogeneous magnetic fields that afford spectral resolution in conventional NMR studies. Given these detrimental conditions, the standard techniques of conventional NMR do not work, and new strategies need to be developed in order to extract valuable information from the NMR signal (*2*–*8*).

Starting from simple relaxation-time measurements, more sophisticated methods of ex situ NMR have been developed, such as Fourier imaging (*5*), velocity imaging (*6*), and multi-

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